Flip-Pushdown Automata

$L(\mathrm{NFPDA}_{\mathrm{fin}})$ ***incomparable to*** GCSL ***and*** CRL

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Introduction: Flip-Pushdown Automaton

Definition 1 (Flip-Pushdown-Automaton NFPDA)

$$A = (Q, \Sigma, \Gamma, \delta, \Delta, q_0, \perp, Q_{\text{fin}})$$

Q *finite set of states, $q_0 \in Q$ start, $Q_{\text{fin}} \subseteq Q$ final states*

Σ *finite input alphabet (symbols on input tape)*

Γ *finite pushdown alphabet, $\perp \in \Gamma$ bottom-of-pushdown symbol*

$\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to Q \times \Gamma^*$ *transition function*

$\Delta : Q \to 2^Q$ *nondeterministic flip transition function*

Introduction: Flip-Pushdown Automaton

Definition 2 (two transitions of NFPDAs)
Let $a \in \Sigma \cup \{\epsilon\}$, $w \in \Sigma^*$, $\gamma, \beta \in \Gamma^*$, $Z \in \Gamma$.

A *chooses nondeterministically between*

$(q, aw, \gamma Z) \vdash_A (p, w, \gamma\beta)$ *if* $(p, \beta) \in \delta(q, a, Z)$

or

$(q, aw, \perp\gamma) \vdash_A \left(p, aw, \perp\gamma^R\right)$ *if* $p \in \Delta(q)$

The reflexive transitive closure of $\vdash_A$ *is* $\vdash_A^*$.

Introduction: Flip-Pushdown Automaton

Example 1 ($L_{\text{copy}} = \{ww \mid w \in \{a,b\}^*\}$)

$$A = (\{q_0, q_1\}, \{a, b\}, \{A, B, \bot\}, \delta, \Delta, q_0, \bot, \emptyset)$$

1. $\delta(q_0, a, \bot) = \{(q_0, \bot A)\}$
2. $\delta(q_0, b, \bot) = \{(q_0, \bot B)\}$
3. $\delta(q_0, a, A) = \{(q_0, AA)\}$
4. $\delta(q_0, b, A) = \{(q_0, AB)\}$
5. $\delta(q_0, a, B) = \{(q_0, BA)\}$
6. $\delta(q_0, b, B) = \{(q_0, BB)\}$
7. $\delta(q_1, a, A) = \{(q_1, \epsilon)\}$
8. $\delta(q_1, b, B) = \{(q_1, \epsilon)\}$
9. $\delta(q_1, \epsilon, \bot) = \{(q_1, \epsilon)\}$
10. $\Delta(q_0) = \{q_1\}$

Introduction: Flip-Pushdown Automaton

Definition 3 (language accepted by final state with k flips)

$$T_k(A) = \{w \in \Sigma^* \mid (q_0, w, \perp) \vdash_A^* (q, \epsilon, \gamma)$$

with exactly k pushdown reversals, for any $\gamma \in \Gamma^*$ and $q \in Q_{\text{fin}}\}$

Definition 4 (language accepted by empty pushdown with k flips)

$$N_k(A) = \{w \in \Sigma^* \mid (q_0, w, \perp) \vdash_A^* (q, \epsilon, \epsilon)$$

with exactly k pushdown reversals, for any $q \in Q\}$

Theorem 1 ($A_1, A_2 \in \mathrm{NFPDA}$)
L, e. g. $N_k(A_1)$, is accepted by A_1 with empty pushdown, iff L, e. g. $T_k(A_2)$, is accepted by A_2 with final state.

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Definition 5

$$L(\mathrm{NFPDA}_k) = \{L \mid L \textit{ accepted by } \mathrm{NFPDA} \textit{ with exactly } k \textit{ flips}\}$$

Definition 6

$$L(\mathrm{NFPDA}_{\mathrm{fin}}) = \bigcup_{k=0}^{\infty} L(\mathrm{NFPDA}_k)$$

Theorem 2 (Sarkar, 2001)

$$\mathrm{CFL} = L(\mathrm{NFPDA}_0) \subseteq L(\mathrm{NFPDA}_1) \subseteq \cdots$$
$$\cdots \subseteq L(\mathrm{NFPDA}_{\mathrm{fin}}) \subseteq L(\mathrm{NFPDA}) = \mathrm{RE}$$

FPDA: Input-Reversal Technique (1)

Theorem 3 (Holzer and Kutrib, 2002**)**
Language $L = N_{k+1}(A_1)$ *is accepted by* $A_1 \in \mathrm{NFPDA}_{k+1}$*, if and only if*

$$L_R = \{wv^R \mid (q_0, w, \perp) \vdash^*_{A_1} (q_1, \epsilon, \perp\gamma)$$

with exactly k *reversals,* $q_2 \in \Delta(q_1)$*, and*

$$(q_2, v, \perp\gamma^R) \vdash^*_{A_1} (q_3, \epsilon, \epsilon)$$

without any reversal$\}$

is accepted by $A_2 \in \mathrm{NFPDA}_k$*, i. e.* $L_R = N_k(A_2)$*.*
(Theorem remains valid for acceptance by final state.)

FPDA: Input-Reversal Technique (2)

Example 2

$$(q_0, abcdefg, \bot) \overset{\text{push}}{\vdash} (q_1, bcdefg, \bot A) \overset{\text{push}}{\vdash} (q_2, cdefg, \bot AB) \overset{\text{flip}}{\vdash}$$
$$(q_3, cdefg, \bot BA) \overset{\text{push}}{\vdash} (q_4, defg, \bot BAC) \overset{\text{pop}}{\vdash} (q_5, efg, \bot BA) \overset{\text{pop}}{\vdash}$$
$$(q_6, fg, \bot B) \overset{\text{pop}}{\vdash} (q_7, g, \bot) \overset{\text{pop/accept}}{\vdash} (q_8, \epsilon, \epsilon)$$

pushdown sequence (after flip): $\bot BA \cdot CC^{-1}A^{-1}B^{-1}\bot^{-1}$

‘reverse-inverse‘ sequence for backward simulation: $w \cdot v^R$

1. $\bot AB \cdot \bot BACC^{-1}$ *(move* $\bot$ *to the end)*
2. $\bot AB \cdot BACC^{-1}\bot$ *(mark symbols, that are touched after last flip)*
3. $\bot AB \cdot \bar{B}\bar{A}\bar{C}\bar{C}^{-1}\bar{\bot}$ *(* $XX^{-1} \to \epsilon$, $X\bar{X}\bar{Y} \to \bar{Y}$, $\bar{X}\bar{X}^{-1} \to \epsilon$, $\bot\bar{\bot} \to \epsilon$ *)*

FPDA: Input-Reversal Technique (3)

ordinary FPDA_{k+1} $A_1 = (Q, \Sigma, \Gamma, \delta, \Delta, q_0, \perp, \emptyset)$

normal form $\forall (p, \gamma) \in \delta(q, a, Z)\ :\ \gamma \in \{\epsilon\} \cup \{ZX \mid X \in \Gamma\}$

generalized FPDA_k $A_2 = (Q \cup \bar{Q} \cup \{\bar{q}_f\}, \Sigma, \Gamma \cup \bar{\Gamma} \cup Q, \delta', \Delta', q_0, \perp, \{\bar{q}_f\})$

finite mapping (pushdown window) $\delta\ :\ Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma^* \to Q \times \Gamma^*$

1. $\forall q \in Q, a \in \Sigma \cup \{\epsilon\}, Z \in \Gamma\ :\ \delta'(q, a, Z)$ contain all $\delta(q, a, Z)$
2. $\forall q \in Q\ :\ \Delta'(q)$ contain all $\Delta(q)$
3. $\forall r \in Q\ :$ if $\Delta(r) \neq \emptyset,$ then $\delta'(r, a, Z)$ contains $(\bar{q}, Z{\perp}r\bar{\perp})$,
   where $q \in Q$ satisfies $(p, \epsilon) \in \delta(q, a, \perp)$ for some $p \in Q$ and $a \in \Sigma \cup \{\epsilon\}$
4. $\forall p, q \in Q, a \in \Sigma \cup \{\epsilon\},$ and $X, Y \in \Gamma\ :$
   $\delta'(\bar{q}, a, \bar{X}\bar{Y})$ contains $(\bar{p}, \bar{X})$ if $(q, XY) \in \delta(p, a, X)$
5. $\forall p, q, r \in Q, a \in \Sigma \cup \{\epsilon\},$ and $X, Y \in \Gamma\ :$
   (a) $\delta'(\bar{q}, a, \bar{X})$ contains $(\bar{p}, \bar{X}\bar{Y})$ if $(q, \epsilon) \in \delta(p, a, Y)$
   (b) $\delta'(\bar{q}, a, Xr\bar{X})$ contains $(\bar{p}, r\bar{Y})$ if $(q, \epsilon) \in \delta(p, a, Y)$
6. $\forall X \in \Gamma, p \in \Delta(r),$ for some $r \in Q\ :\ \delta'(\bar{p}, \epsilon, {\perp}Xr\bar{X})$ contains $(\bar{q}_f, \epsilon)$

FPDA: Input-Reversal Technique (4)

transitions of A_2 for backward simulation of A_1

forward		backward	
flip	$p \in \Delta(q)$	$(\bar{p}', Z\bot q\bar{\bot}) \in \delta'(\bar{q}, a, Z)$ if $(p'', \epsilon) \in \delta(p', a, \bot)$	push
push	$(p, XY) \in \delta(q, a, X)$	$(\bar{q}, \bar{X}) \in \delta'(\bar{p}, a, \bar{X}\bar{Y})$	pop
pop	$(p, \epsilon) \in \delta(q, a, Y)$	$(\bar{q}, \bar{X}\bar{Y}) \in \delta'(\bar{p}, a, \bar{X})$	a-push
		$(\bar{q}, r\bar{Y}) \in \delta'(\bar{p}, a, Xr\bar{X})$	b-push
accept	$(p, \epsilon) \in \delta(q, a, \bot)$	–	
–		$(\bar{q}_f, \epsilon) \in \delta'(\bar{p}, \epsilon, \bot Xq\bar{X})$ if $p \in \Delta(q)$	accept

FPDA: Unary Languages, Strict Hierarchy

Corollary 1 (Holzer and Kutrib, 2002**)**
If L is a unary language accepted by a NFPDA_k for some $k \geq 0$, then L is a regular language.

Theorem 4 (Holzer and Kutrib, 2002**)**

$$\forall k \geq 0 \ : \ L(\mathrm{NFPDA}_k) \subset L(\mathrm{NFPDA}_{k+1})$$

$$\forall k \geq 0 \ : \ L(\mathrm{DFPDA}_k) \subset L(\mathrm{DFPDA}_{k+1}),$$

$$L_k = \{\#w_1\$w_1\#w_2\$w_2\# \cdots \#w_k\$w_k\# \mid w_i \in \{a,b\}^* \text{ for } 1 \leq i \leq k\}$$

FPDA: Closure Properties, full TRIO

	CFL			RE
	$L(\text{NFPDA}_0)$	$L(\text{NFPDA}_k)$	$L(\text{NFPDA}_{\text{fin}})$	$L(\text{NFPDA})$
$\cup$	$\surd$	$\surd$	$\surd$	$\surd$
$\cap$	–	–	–	$\surd$
$^-$	–	–	–	–
$\cdot$	$\surd$	–	$\surd$	$\surd$
$*$	$\surd$	–	–	$\surd$
h	$\surd$	$\surd$	$\surd$	$\surd$
h^{-1}	$\surd$	$\surd$	$\surd$	$\surd$
$\cap_{\text{REG}}$	$\surd$	$\surd$	$\surd$	$\surd$

FPDA: Computational Complexity

Theorem 5 (Holzer and Kutrib, 2002**)**

$$\mathrm{CFL} \subset L(\mathrm{NFPDA}_k) \subset \mathrm{CSL} \text{ for } k \geq 1$$

Theorem 6 (Holzer and Kutrib, 2002**)**
For languages in $L(\mathrm{NFPDA}_k)$:

1. *Fixed membership problem is $\mathrm{LOG(CFL)}$-complete.*
2. *General membership problem is P-complete.*

FPDA: Open and solved Problems

- deterministic $L(\mathrm{DFPDA}_k)$ hierarchy [HKu03]
  1. $N_{=k}(\mathrm{DFPDA}) = N_{\leq k}(\mathrm{DFPDA}) \subset T_{=k}(\mathrm{DFPDA}) \subset T_{\leq k}(\mathrm{DFPDA})$
  2. $L(\mathrm{DFPDA}_{=k}) \subset L(\mathrm{DFPDA}_{\leq k}) \subset L(\mathrm{NFPDA}_k)$
- relationship to other language classes:
  1. $L(\mathrm{NFPDA}_{\mathrm{fin}})$ maybe incomparable to $\mathrm{E0L}$ or $\mathrm{ET0L}$
     (a) $\mathrm{E0L} \ni \{a^n b^n c^n \mid n \geq 0\} \notin \mathrm{NFPDA}_{\mathrm{fin}}$
     (b) looking for: semi-linear language $\notin \mathrm{ET0L}$
  2. $L(\mathrm{NFPDA}_{\mathrm{fin}})$ is incomparable to GCSL and CRL

$L(\mathrm{NFPDA_{fin}})$ **incomparable to** GCSL **and** CRL

Corollary 2
$L(\mathrm{NFPDA_{fin}})$ *is incomparable to* GCSL *under set inclusion. The same holds for* CRL*.*

$$L_{\mathrm{copy}} \in L(\mathrm{NFPDA_{fin}})$$
$$L_{\mathrm{copy}} \notin \mathrm{GCSL}$$

$$L_{\mathrm{expo}} = \{a^{2^n} \mid n \geq 0\}$$
$$L_{\mathrm{expo}} \notin L(\mathrm{NFPDA_{fin}}),$$ **Corollary 1** $\rightsquigarrow L_{\mathrm{expo}} \notin \mathrm{REG}$
$$L_{\mathrm{expo}} \in \mathrm{GCSL}$$

References

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